

# Inflationary Super-Hubble Waves and the Size of the Universe

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The effect of the scalar spectral index on inflationary super-Hubble waves is to amplify/damp large wavelengths according to whether the spectrum is red ( $n_s < 1$ ) or blue ( $n_s > 1$ ). As a consequence, the large-scale temperature correlation function will unavoidably change sign at some angle if our spectrum is red, while it will always be positive if it is blue. We show that this inflationary filtering property also affects our estimates of the size of the homogeneous patch of the universe through the Grishchuk-Zel'dovich effect. Using the recent quadrupole measurement of ESA's Planck mission, we find that the homogeneous patch of universe is at least 87 times bigger than our visible universe if we accept Planck's best fit value  $n_s = 0.9624$ . An independent estimation of the size of the universe could be used to independently constrain  $n_s$ , thus narrowing the space of inflationary models.

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## I. INTRODUCTION

The temperature power spectrum coefficients of the Cosmic Microwave Background (CMB) radiation field, the  $C_\ell$ s, when plotted as a function of their multipoles  $\ell$ , comprise one of the most famous figures of modern cosmology. In face of the recent and unprecedented boost that this subject has received from Planck's data release [1, 2], CMB will certainly retain its distinguished status among cosmologists during the next coming years. Not so famous, but yet conveying the same information, is the plot of the angular two-point correlation function  $C(\theta)$ :

$$C(\theta) = \langle \Delta T_1 \Delta T_2 \rangle = \sum_{\ell=1}^{\infty} \frac{(2\ell+1) C_\ell}{4\pi} P_\ell(\cos \theta). \quad (1)$$

While either the position space  $C(\theta)$  or the harmonic  $C_\ell$ s are equivalent and sufficient to fully characterize the CMB morphology in the Gaussian and statistically isotropic regime, the latter are mainly used for extracting cosmological information since, in this regime, the  $C_\ell$ s decouple from one another, making the harmonic analysis much more transparent. The angular correlation function in Eq. (1), instead of splitting the information in several independent multipoles, goes on the opposite direction and compresses information from virtually all multipolar scales to give the physics at a single angular separation  $\theta$ . This does not mean that a real space approach to CMB is prohibitive, and indeed non-trivial physics can result from it [3–5]. However, the result of this compression is a featureless and literally monotone function of  $\theta$ , as shown by the continuous line of Fig. (1). Note that, for the sake of exposition, we have included a primordial dipole  $C_1 = 3C_2$  in Eq. (1), with the proportionality constant fixed by the Sachs-Wolfe effect  $\ell(\ell+1)C_\ell = \text{constant}$ .

By eye, the most prominent feature of Fig. (1), if not the only, is that  $C(\theta)$  has a root at  $\theta \simeq 75^\circ$ . The immediate

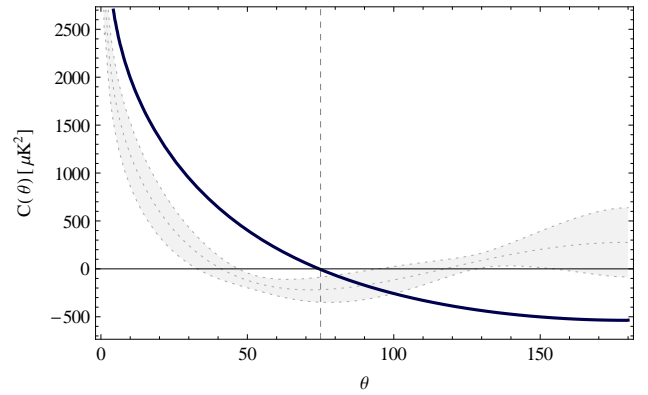


Figure 1: CMB two-point correlation function as a function of angular separation  $\theta$  (solid, thick line) in the  $\Lambda$ CDM model. This curve includes a primordial dipole  $C_1 = 3C_2$ , in accordance with the Sachs-Wolfe effect. The removal of this dipole results in the appearance of two roots, one at  $\theta \approx 40^\circ$  and other at  $\theta \approx 120^\circ$ ; this is shown by the dashed light curve, which also includes cosmic variance errors bars.

question which comes to mind is: why is there an angular scale above which CMB photons become uncorrelated? At these angles, the photons we detect are essentially sourced by plane wave perturbations of the gravitational potential through the Sachs-Wolfe effect. Since these waves have a length comparable to the radius of our particle horizon, each of their crests will subtend an angle of approximately  $90^\circ$  in the CMB sky, while larger waves will only have a fraction of their crests inside the horizon. Therefore, photons separated by angles larger than  $90^\circ$  are, on average, probing the maxima and the minima of these waves. This gives us the anti-correlation pattern observed, since  $C(\theta) = \langle + - \rangle < 0$ . What about larger waves which are shifted to the left/right, so that only their positive/negative crests lies inside the horizon? Photons probing these waves will always be correlated, since  $C(\theta) = \langle ++ \rangle = \langle -- \rangle > 0$ . However, since *a priori* there is no upper limit for the wavelength we should consider, it is conceivable that the correlation function

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could receive an infinite contribution from them, possibly diverging. Unless these waves are weighted in inverse proportion to their wavelengths, in which case the contributions to  $C(\theta)$  from larger waves could saturate at some finite number.

Let us analyze this issue closer by invoking the definition of the real-space temperature correlation function:

$$C(\theta) = \langle \Delta T(\tau_0, \hat{\mathbf{n}}_1), \Delta T(\tau_0, \hat{\mathbf{n}}_2) \rangle.$$

Here,  $\tau_0$  is our present cosmological time,  $\hat{\mathbf{n}}_i$  represents the direction of each incoming photon, and  $\theta$  is the angle between them. For large angles the Sachs-Wolfe effect,  $\Delta T = \Phi/3$  for adiabatic initial conditions, is the dominant contribution to  $C(\theta)$ . Going to Fourier space and using

$$\langle \Phi(\mathbf{k})\Phi(\mathbf{q}) \rangle = 2\pi^2 k^{-3} \mathcal{P}(k) \delta^{(3)}(\mathbf{k} + \mathbf{q}),$$

as expected from the symmetries of a Gaussian, homogeneous and isotropic universe [6], we find after a bit of algebra that

$$C(\theta) = \frac{1}{9} \int_0^\infty \frac{dk}{k} \frac{\sin kr}{kr} \mathcal{P}(k), \quad (2)$$

where  $r = 2\Delta\tau\sqrt{1 - \cos\theta}$  and  $\Delta\tau$  is the radius of the last scattering surface. Essentially all models of the early universe predict a *primordial power spectrum*  $\mathcal{P}(k)$  which is a power law in  $k$ . Without loss of generality this function can be written as

$$\mathcal{P}(k) = Ak^{n_s-1} \quad (3)$$

where  $n_s$  is known as the scalar spectral index and  $A$  is an amplitude to be fixed. Using this power spectrum in Eq. (2) we find that after integration that

$$C(\theta) \propto (1 - \cos\theta)^{\frac{1-n_s}{2}}, \quad 1 < n_s < 3. \quad (4)$$

There are two important features to be noted in this expression. First, the amplitude is a finite function of  $n_s$ . Second, the above correlation function has *no root* in the range  $1 < n_s < 3$  and  $0 \leq \theta \leq 180^\circ$  – see Fig. (2). Indeed, this agrees with our discussion, since for  $n_s > 1$  the power spectrum in Eq. (3) filters out low values of  $k$ , ensuring that arbitrarily large waves will cease to contribute to  $C(\theta)$ . For a Harrison-Zel’dovich spectrum ( $n_s = 1$ ) the above integral has a *finite* dependence on  $\theta$ :

$$C(\theta) \propto \ln \left( \frac{1}{1 - \cos\theta} \right)^{1/2}, \quad n_s = 1. \quad (5)$$

However, the amplitude here is a logarithmic divergent function of  $k$ , as one can check by fixing the lower limit of the integral in Eq. (2) at  $k_0$  and carrying a Taylor expansion around zero. Evidently, this divergent amplitude corresponds to a monopole  $C_0$ , and can be safely regularized. The important point here – and also for the case  $n_s < 1$ , although with Eq. (5) replaced by a more

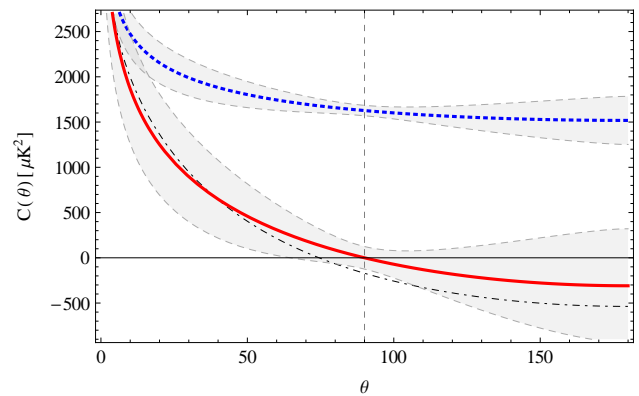


Figure 2: Large-angle temperature correlation function for different values of the scalar spectral index  $n_s$ . The curves represent analytical expressions of  $C(\theta)$  for a Harrison-Zel’dovich ( $n_s = 1$ , continuous line) and blue ( $n_s = 1.2$ , dashed line) power spectra. The envelopes represent  $1\sigma$  cosmic variance error bars. These curves were arbitrarily normalized at  $\theta = 1^\circ$  to match the  $\Lambda$ CDM correlation function (thin, dot-dashed line).

complicated expression – is that the correlation function has *one root* at  $\theta = 90^\circ$  (Fig. (2)). Both the infinite amplitude and the root of this function agrees with the interpretation of  $\mathcal{P}(k)$  acting as a constant filter which allows arbitrarily large waves to contribute to  $C(\theta)$  if  $n_s = 1$ .

Faced with this filtering property, another important question is related to the use we can make of it. Unfortunately, using the real-space behavior of  $C(\theta)$  to constrain  $n_s$  would be a real challenge. Not only the errors in the measurement of  $C(\theta)$  at large angles are strongly correlated, but the observed  $C(\theta)$  is known to suffer from a coherent lack of power at these scales [7–9] (see also Ref. [10] for a recent review of this and other related large-angle anomalies.) We are not going to pursue these issues here. Instead, we note that the above example poses a different but related question: how far beyond the radius of the observable universe can a wave affect CMB, for a given  $n_s$ ? Since for a blue spectrum ( $n_s > 1$ ) arbitrarily large waves will contribute less and less to the temperature fluctuation in contrast to a red one ( $n_s < 1$ ), we can expect the former contribution to saturate at a larger maximum wavelength than the later. Moreover, by restricting the amplitude of these waves to the linear sector of perturbation theory, this effect provides a natural mechanism in terms of which we can put lower bounds on the extension of the homogeneous patch of the universe.

The idea of using large (super-Hubble) waves to estimate the radius of the homogeneous universe is not new and was first proposed by Grishchuk and Zel’dovich (henceforth refereed to as the GZ effect) [11]. In Ref. [12], the effect of super-Hubble linear perturbations from a scalar field was investigated, where it was found that the linear universe is at least 45 times larger than the ob-

servable universe. In Ref. [13] a single sinusoidal was employed, leading to a lower bound on the radius of the homogeneous universe of 65 times the present particle horizon. Further extensions were pursued in order to apply the GZ effect to spatially curved universes [14], and in the estimation of possible inflationary remnants in CMB [15]. However, in all of these approaches, and also in the original GZ paper, the effect of the scalar spectral index was completely ignored. Instead, the “wall” of the homogeneous universe was assumed to be well characterized by a sharp feature in the power spectrum:  $P(k) = A\delta(k - k_{GZ})$ . Since the original GZ effect was proposed before the formulation of modern inflationary models, it is reasonable that one should model a break in the scale of homogeneity by a sharp peak in the spectrum. From this point of view, the approach we will follow here is a natural extension of this idea. Given that inflation predicts the existence of filtered stochastic waves, as we illustrated above, this very property will dictate the limits on the size of the homogeneous universe that current CMB data can place. In a sense, our proposal is to estimate the radius of the universe by measuring it from inside-out, rather than at the edge.

This work will be organized as follows. In Sec. II we will define precisely what we mean by filtered stochastic waves, and show how they should be normalized on arguments of global scale invariance. In Sec. III we restrict attention to super-Hubble waves and calculate the impact of these waves on the low multipole moments of CMB temperature map. Finally, we use the recently released Planck data to put constraints on the size of the homogeneous universe in Sec. IV. We conclude in Sec. V with our final remarks and some further perspectives of development. Throughout the text, we use natural units in which  $c = 1$ . Our convention for the Fourier transform is to democratically distribute factors of  $(2\pi)^{3/2}$  among integrals.

## II. FILTERED STOCHASTIC WAVES

We will start by considering the standard scenario of scalar and adiabatic perturbations evolving in a spatially flat, Friedmann-Lemaître-Robertson-Walker (FLRW) universe dominated by cold dark matter, baryons, radiation and a cosmological constant. In the absence of anisotropic stress, the spacetime metric is

$$ds^2 = a^2(\tau) [-(1 + 2\Phi)d\tau^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j]$$

where  $\Phi(\tau, \mathbf{x})$  is the gauge-invariant gravitational potential. Usually, linear perturbation theory is carried in Fourier space, where  $\Phi(\tau, \mathbf{k})$  is the relevant quantity. In the standard FLRW case, we can split the functional dependence of this function as a dependence on the initial conditions due to inflation,  $\psi(\mathbf{k})$ , times the dependence on the dynamical evolution due to gravity,  $\phi(\tau, k)$ :

$$\Phi(\tau, \mathbf{k}) = \phi(\tau, k)\psi(\mathbf{k}).$$

Note that the dependence on the direction of  $\mathbf{k}$  appears even in an isotropic background setup, although here it comes exclusively from the initial conditions  $\psi$ , and not from the spacetime dynamics as would happen, for example, in anisotropic universes [16, 17]. The function  $\phi$  can develop a  $k$ -dependence only through the Laplacian appearing in the evolution equation [18]:

$$\phi'' + 3\mathcal{H}(1 + c_s^2)\phi' + [2\mathcal{H}' + \mathcal{H}^2(1 + 3c_s^2)]\phi + k^2 c_s^2 \phi = 0 \quad (6)$$

where  $c_s^2$  is the sound speed of whatever fluids constitute the universe,  $\mathcal{H}$  is the conformal Hubble parameter and the primes mean derivatives with respect to conformal time  $\tau$ . The general idea behind inflationary models is that inflation has lasted for a finite number of e-folds, stretching an initially homogeneous patch of the universe of typical size  $H^{-1}$  up to some radius  $L$ . During this evolution, the dynamics of  $\phi$  can be linked to that of the curvature perturbations  $\zeta$ , which is known to be constant at super-Hubble scales. Since in this work we shall be mainly interested in scales where  $k \ll \mathcal{H}$ , we will make the mildly technical assumption that  $\phi$  does not depend on  $k$  at all, that is, we assume that

$$\Phi(\tau, \mathbf{k}) = \phi(\tau)\psi(\mathbf{k}) \quad (7)$$

is a good description for the gravitational potential at all wavelengths. This is certainly not valid at small scales during, let's say, the radiation dominated phase, where  $\phi$  is an oscillating and decaying function of  $k$ . However, during inflation, every factor of  $k$  in the solution of Eq. (6) is accompanied by a factor of  $1/a$ . So, although we will integrate Eq. (7) over all values of  $\mathbf{k}$ , it is expected that the corrections to the  $k$ -dependence of  $\phi$  are suppressed by the quasi-exponential expansion of the inflationary universe [19]. In conclusion, Eq. (7) ensures that  $\Phi(\tau, \mathbf{x})$  is also a separable function of  $\tau$  and  $\mathbf{x}$ , which we can always decompose as

$$\Phi(\tau, \mathbf{x}) = \phi(\tau) \sum_{\ell, m} \varpi_{\ell m}(x) Y_{\ell m}(\hat{\mathbf{n}}). \quad (8)$$

Here, the multipolar coefficients  $\varpi_{\ell m}(x)$  are understood as complex Gaussian random variables satisfying

$$\langle \varpi_{\ell m}(x) \rangle = 0, \quad \langle \varpi_{\ell m}(x) \varpi_{\ell' m'}^*(x) \rangle = |\varpi_{\ell}(x)|^2 \delta_{\ell \ell'} \delta_{m m'},$$

where, it should be noted, the averages are taken at a fixed radius  $x$ . This ensures that the gravitational potential in Eq. (8) is also a Gaussian random variable whose two-point correlation function is given by an expression similar to Eq. (1). In what follows, we will rewrite the multipoles  $\varpi_{\ell m}(x)$  as

$$\varpi_{\ell m}(x) = \varpi_{\ell}(x) \varphi_{\ell m} \quad (9)$$

where  $\varphi_{\ell m}$  is a (complex) Gaussian variable satisfying

$$\langle \varphi_{\ell m} \varphi_{\ell' m'}^* \rangle = \delta_{\ell \ell'} \delta_{m m'}. \quad (10)$$

This decomposition is convenient because it separates the gravitational power spectrum  $\varpi_\ell(x)$ , which is a fixed physical quantity, from the purely random phases  $\varphi_{\ell m}$ . This splitting is always possible in a FLRW background universe, where CMB is statistically isotropic.

Our task now is to relate the real-space spectrum  $\varpi_\ell(x)$  to the scalar spectral index  $n_s$ . In order to proceed, note that if we decompose Eq. (7) similarly to what was done in Eqs. (8) and (9), we can relate  $\varpi_\ell(x)$  to its Fourier dual,  $\varpi_\ell(k)$ , through the so-called Hankel transform [4]:

$$\varpi_\ell(x) = \sqrt{\frac{2}{\pi}} i^\ell \int_0^\infty dk k^2 j_\ell(kx) \varpi_\ell(k). \quad (11)$$

The Fourier-space spectrum  $\varpi_\ell(k)$  is related to the initial conditions  $\psi(\mathbf{k})$ , which are also Gaussian random variables satisfying

$$\langle \psi(\mathbf{k}) \psi(\mathbf{q}) \rangle = P(k) \delta^{(3)}(\mathbf{k} + \mathbf{q}), \quad P(k) = 2\pi^2 A k^{n_s-4}. \quad (12)$$

Writing a simple expression for  $\psi(\mathbf{k})$  is virtually impossible since, for each wavenumber  $\mathbf{k}$ , inflation has produced different initial conditions with independent phases and amplitudes. This is confirmed by the simple observation that galaxies tend to cluster differently at different scales and in different directions. Since we want to place bounds on the size of the homogeneous patch of the universe, we can speculate that a *typical realization* of the initial conditions at super-Hubble scales is

$$\psi_{\mathbf{k}} = \sqrt{P(k)} e^{-i\mathbf{k} \cdot \mathbf{L}} \quad (13)$$

where  $L = |\mathbf{L}|$  is the radius of the homogeneous patch. Physically, this ansatz represents a standing wave whose amplitude is filtered by the square root of the variance  $P(k)$  – we call it a filtered stochastic wave. We stress that Eq. (13) is an ansatz for *one realization* of  $\psi$  at a fixed wavenumber  $\mathbf{k}$ , not the full Gaussian field. Nonetheless, this ansatz is quite general because we can always decompose  $\psi(\mathbf{k})$  as  $\sqrt{P(k)} \tilde{\psi}(\mathbf{k})$ , and write  $\tilde{\psi}(\mathbf{k})$  as a Fourier series of plane waves  $e^{-i\mathbf{k} \cdot \mathbf{L}}$ . Evidently, the fact that different waves have different phases and amplitudes has to be taken into account. This can be easily implemented in harmonic (rather than in Fourier) space by defining  $\varpi_\ell(k)$  as the angular coefficients of  $\psi_{\mathbf{k}}$ <sup>1</sup>

$$\sqrt{P(k)} e^{-i\mathbf{k} \cdot \mathbf{L}} \equiv \sum_\ell (2\ell + 1) \varpi_\ell(k) P_\ell(\cos \theta)$$

from where it follows that

$$\varpi_\ell(k) = (-i)^\ell \sqrt{P(k)} j_\ell(kL). \quad (14)$$

Then, since we are working under the hypothesis of statistical isotropy, the randomness of phases and amplitudes can be implemented by multiplying  $\varpi_\ell(k)$  by the

standard Gaussian variable  $\varphi_{\ell m}$ . This is achieved automatically by Eqs. (9) and (11).

We can now compute the integral in Eq. (11) using Eq. (14). Since by hypothesis  $x \ll L$ , the result can be expanded in powers of  $x/L$ . We find

$$\varpi_\ell(x) = g_\ell(n_s) \epsilon^\ell + \mathcal{O}(\epsilon^{\ell+2}), \quad (15)$$

where  $\epsilon \equiv x/L \ll 1$  and

$$g_\ell(n_s) \equiv 2^{\frac{n_s}{2}-1} \pi^{3/2} \frac{\Gamma(\ell + \frac{2+n_s}{4})}{\Gamma(\ell + \frac{3}{2}) \Gamma(\frac{4-n_s}{4})}. \quad (16)$$

This expression, together with Eq. (9) and the solution of Eq. (6), completes the construction of the gravitational potential at large scales:

$$\Phi(\tau, \mathbf{x}) = \phi(\tau) \sum_{\ell, m} g_\ell(n_s) \varphi_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}) \epsilon^\ell. \quad (17)$$

As a last comment, note that, in order to obtain Eq. (15), we have fixed the amplitude in Eq. (12) as  $A \equiv L^{n_s+2}$ . This choice ensures that the gravitational potential is a homogeneous function of  $x$  and  $L$ , that is,  $\Phi(x; L) = \Phi(\lambda x; \lambda L)$  for some real  $\lambda$ . This is necessary because  $x$  and  $L$  are comoving coordinates, so that their scalings are meaningless. Incidentally, this prescription does not fix  $A$  uniquely, since the choice  $A = B L^{n_s+2}$  would also work for any constant  $B$ . However, this new constant can be absorbed in the initial value of  $\phi(\tau)$  entering in Eq. (17), which will be properly normalized at CMB decoupling later on.

### III. LARGE SCALE TEMPERATURE FLUCTUATIONS

Moving forward, we will now compute the contribution of the large scale gravitational potential in Eq. (17) to the low multipoles of CMB. The main contribution to CMB at large scales can be accounted for through the well-known Sachs-Wolfe formula [20]:

$$\Delta T(\tau_0, \hat{\mathbf{n}}) = \left[ \frac{1}{4} \delta_r + \Phi \right] (\tau, \mathbf{x}_{\text{dec}}) + 2 \int_{\tau_{\text{dec}}}^{\tau_0} \frac{\partial \Phi(\tau, \mathbf{x}(\tau))}{\partial \tau} d\tau + \hat{\mathbf{n}} \cdot \nabla v|_{\tau_{\text{dec}}}^{\tau_0}, \quad (18)$$

where

$$\mathbf{x}(\tau) = (\tau_0 - \tau) \hat{\mathbf{n}}.$$

Since at CMB decoupling the gravitational potential is  $\Phi(\tau_{\text{dec}}, \mathbf{x}_{\text{dec}})$ , we will present our results in powers of

$$\epsilon_{\text{dec}} \equiv \frac{x_{\text{dec}}}{L} \ll 1,$$

where  $x_{\text{dec}} \equiv |\mathbf{x}(\tau_{\text{dec}})|$ . In order to calculate the multipolar decomposition of Eq. (18) we will need to solve

<sup>1</sup> Note that these coefficients can also be interpreted as the angular average of Eq. (13):  $\varpi_\ell(k) = \frac{1}{2} \int_{-1}^1 d\mu \psi_{\mathbf{k}} P_\ell(\mu)$ .

Eq. (6) numerically, together with the background Friedmann equations. We have chosen a fiducial  $\Lambda$ CDM model with best fit parameters given by Planck data alone [1], which are:  $\Omega_m = 0.3175$ ,  $z_{\text{eq}} = 3402$  and  $z_{\text{dec}} = 1090.43$ . To solve Eq. (6) we use adiabatic initial conditions such that

$$\phi(0) = \phi_p, \quad \phi'(0) = 0, \quad (19)$$

where  $\phi_p$  is the primordial value of  $\phi(\tau)$ . As we are going to see, all contributions to  $\Delta T/\phi(\tau_{\text{dec}})$  depend on ratios like  $\phi(\tau)/\phi(\tau_{\text{dec}})$  which, at large scales, are independent of the choice of the value of  $\phi_p$ . On the other hand, the absolute value of  $\phi(\tau_{\text{dec}})$  does influence our estimates of  $\Delta T$ , and so it will have to be fixed. We fix this value by restricting our analysis to the linear domain of perturbation theory at CMB decoupling, that is, we impose the condition

$$|\delta\rho/\rho|_{\tau_{\text{dec}}} \lesssim 1. \quad (20)$$

At large scales, the relation between the energy density perturbation and the gravitational potential is

$$4\pi G a^2 \delta\rho \approx -3\mathcal{H}(\phi' + \mathcal{H}\phi),$$

where we have used the separability condition in Eq. (7) to isolate the spatial dependence of  $\delta\rho$ . Changing variables from  $\tau$  to  $a$  and using  $3\mathcal{H}^2 = 8\pi G \rho a^2$ , the time dependence of the fractional energy density becomes

$$\frac{\delta\rho}{\rho} = -2 \frac{d}{da} (a\phi).$$

Defining  $\phi(a) = \phi_p f(a)$  such that  $f(0) = 1$  and  $f'(0) = 0$ , we find numerically that  $|\delta\rho/\rho|_{\tau_{\text{dec}}} = 1.828 \phi_p$ . The condition in Eq. (20) then fixes  $\phi(\tau_{\text{dec}})$  uniquely as

$$\phi(\tau_{\text{dec}}) = 0.512, \quad (21)$$

which, in turn, fixes the ratio  $L/x_{\text{dec}}$ , as we shall see.

### A. Sachs-Wolfe effect

The Sachs-Wolfe (SW) effect is the simplest contribution to compute. First, note that at super-Hubble scales the radiation plasma has a negligible peculiar speed, which implies that  $\delta'_r - 4\Phi' = 0$  [18]. Restricting our analysis to adiabatic initial conditions, for which  $\delta_r(0, \mathbf{x}) = -2\Phi(0, \mathbf{x})$ , it then follows that

$$\frac{1}{4}\delta_r(\tau_{\text{dec}}, \mathbf{x}_{\text{dec}}) + \Phi(\tau_{\text{dec}}, \mathbf{x}_{\text{dec}}) = \Phi(\tau_{\text{dec}}, \mathbf{x}_{\text{dec}})\mathcal{N},$$

where  $\mathcal{N} \equiv 2 - 3\phi(0)/2\phi(\tau_{\text{dec}})$  needs to be evaluated numerically. We find

$$\mathcal{N} = \frac{1}{3} + 0.0644 \approx 0.4.$$

Using Eq. (17), the Sachs-Wolfe temperature fluctuation is then written as

$$\Delta T_{\text{SW}}(\tau_0, \hat{\mathbf{n}}) = \phi(\tau_{\text{dec}}) \sum_{\ell, m} \varphi_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}) \mathcal{S}_{\ell} \epsilon_{\text{dec}}^{\ell}, \quad (22)$$

where

$$\mathcal{S}_{\ell} \equiv g_{\ell}(n_s) \mathcal{N}. \quad (23)$$

### B. Integrated Sachs-Wolfe effect

In order to compute the contribution from the integrated Sachs-Wolfe (ISW) effect we use the identity

$$\dot{\Phi}(\tau, \mathbf{x}(\tau)) = \frac{\partial \Phi}{\partial \tau} - \hat{\mathbf{n}} \cdot \nabla \Phi$$

where dot means total derivative with respect to  $\tau$  and where we have used  $d\mathbf{x}/d\tau = -\hat{\mathbf{n}}$ . Plugging this in the ISW expression we readily find

$$\Delta T_{\text{ISW}}(\tau_0, \hat{\mathbf{n}}) = 2 \Phi|_{\tau_{\text{dec}}}^{\tau_0} + 2 \int_{\tau_{\text{dec}}}^{\tau_0} \hat{\mathbf{n}} \cdot \nabla \Phi d\tau.$$

If we align  $\mathbf{L}$  with the  $z$ -axis and write  $\mathbf{x}$  in spherical coordinates, then  $\hat{\mathbf{n}} \cdot \nabla = \partial/\partial x$  and we have

$$\hat{\mathbf{n}} \cdot \nabla \Phi = \phi(\tau) \sum_{\ell, m} g_{\ell} \varphi_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}) \frac{d}{dx} \epsilon^{\ell}.$$

Joining all the terms and dropping the monopole  $2\Phi(\tau_0, 0)$ , we arrive at

$$\Delta T_{\text{ISW}}(\tau_0, \hat{\mathbf{n}}) = \phi(\tau_{\text{dec}}) \sum_{\ell, m} \varphi_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}) \mathcal{I}_{\ell} \epsilon_{\text{dec}}^{\ell} \quad (24)$$

where

$$\mathcal{I}_{\ell} \equiv -2g_{\ell}(n_s) \left[ 1 - \int_{\tau_{\text{dec}}}^{\tau_0} \frac{\phi(\tau)}{\phi(\tau_{\text{dec}})} \frac{d}{dx} \left( \frac{x}{x_{\text{dec}}} \right)^{\ell} d\tau \right]. \quad (25)$$

### C. Doppler effect

At the scales we are interested, the mean free path of photons is very small, and we can think of baryons and radiation as tightly coupled into a single fluid (tight-coupling limit). The Doppler effect at direction  $\hat{\mathbf{n}}$  then involves the baryon-radiation plasma peculiar velocity  $\mathbf{v} = \nabla v$  through  $\hat{\mathbf{n}} \cdot \nabla v|_{\tau_{\text{dec}}}^{\tau_0}$ . The relation between  $v$  and the gravitational potential follows from Einstein's equation

$$\frac{\partial}{\partial \tau} (a\Phi) = -4\pi G a^3 \rho (1 + \omega) v$$

where  $\rho = \rho_m + \rho_r$ , and  $\omega$  is the effective equation of state of the plasma. Since in our approximation  $\Phi(\tau, \mathbf{x}) = \phi(\tau)\psi(\mathbf{x})$ , the velocity can be rewritten as

$$v = f(\tau) \sum_{\ell, m} g_{\ell} \varphi_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}) \epsilon^{\ell}.$$

with  $f(\tau) = -\frac{d}{d\tau}(a\phi)/4\pi G a^3 \rho(1+\omega)$ . Using  $\hat{\mathbf{n}} \cdot \nabla v = \partial v / \partial x$ , we find

$$\Delta T_{\text{Dop}}(\tau_0, \mathbf{x}_{\text{dec}}) = \phi(\tau_{\text{dec}}) \sum_{\ell, m} \varphi_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}) \mathcal{D}_{\ell} \epsilon_{\text{dec}}^{\ell}, \quad (26)$$

where we have introduced

$$\mathcal{D}_{\ell} \equiv \frac{g_{\ell}(n_s)}{x_{\text{dec}} \phi(\tau_{\text{dec}})} [f(\tau_0) \delta_{\ell 1} - f(\tau_{\text{dec}}) \ell]. \quad (27)$$

#### IV. OBSERVATIONAL CONSTRAINTS

We will now use the recently released Planck data to put constraints on the ratio  $L/x_{\text{dec}}$ . With the relevant effects computed, we can gather all contributions to  $\Delta T(\tau_0, \hat{\mathbf{n}})$  in the simpler expression

$$\Delta T(\tau_0, \hat{\mathbf{n}}) = \sum_{\ell, m} a_{\ell m}(\tau_0) Y_{\ell m}(\hat{\mathbf{n}}), \quad (28)$$

with the multipolar coefficients given by

$$a_{\ell m}(\tau_0) = \phi(\tau_{\text{dec}}) \varphi_{\ell m} (\mathcal{S}_{\ell} + \mathcal{I}_{\ell} + \mathcal{D}_{\ell}) \epsilon_{\text{dec}}^{\ell}. \quad (29)$$

This expression, which is valid up to terms of order  $\epsilon_{\text{dec}}^{\ell+2}$ , shows that the stronger constraint on the ratio  $L/x_{\text{dec}}$  will come from the lower CMB multipoles. Since the monopole  $C_0$  is unobservable, the first contribution comes from the intrinsic CMB dipole. However, in what concerns super-Hubble waves, the intrinsic dipole produced by Eqs. (22) and (24) is exactly canceled, at lower order in  $\epsilon_{\text{dec}}$ , by the Doppler dipole in Eq. (26). This can be seen in the first panel of Fig. (3), where we show some plots of the first multipoles  $\Delta T_{\ell}$  entering the total temperature fluctuation, and defined by

$$\Delta T_{\ell}(\theta; n_s) \equiv (\mathcal{S}_{\ell} + \mathcal{I}_{\ell} + \mathcal{D}_{\ell}) Y_{\ell 0}(\theta). \quad (30)$$

The dipole cancellation is a peculiar feature of the GZ effect which happens quite generally in a  $\Lambda$ CDM universe – see Ref. [13] for an explicit demonstration. Here we find that it also holds, regardless of  $n_s$ . As a general behavior, we can see that the amplitude of the other multipoles grow with growing values of  $n_s$ .

It's also instructive to see the individual contributions from the SW, ISW and Doppler effects to the multipoles in Eq. (30). These are shown separately in each panel of Fig. (4). Although not explicit, one can infer from the first panel in this figure that the sum of the Doppler and ISW contributions exactly cancels the SW term.

In order to proceed, note that, since  $\varphi_{\ell m}$  are standard Gaussian variables, it follows from Eq. (29) that

$$C_{\ell} = \phi(\tau_{\text{dec}})^2 (\mathcal{S}_{\ell} + \mathcal{I}_{\ell} + \mathcal{D}_{\ell})^2 \epsilon_{\text{dec}}^{2\ell}.$$

Using Planck's best fit value  $n_s = 0.9624$  and the linearity prescription of Eq. (21), the quadrupole amplitude evaluates numerically to

$$C_2 = 0.067 \left( \frac{x_{\text{dec}}}{L} \right)^4. \quad (31)$$

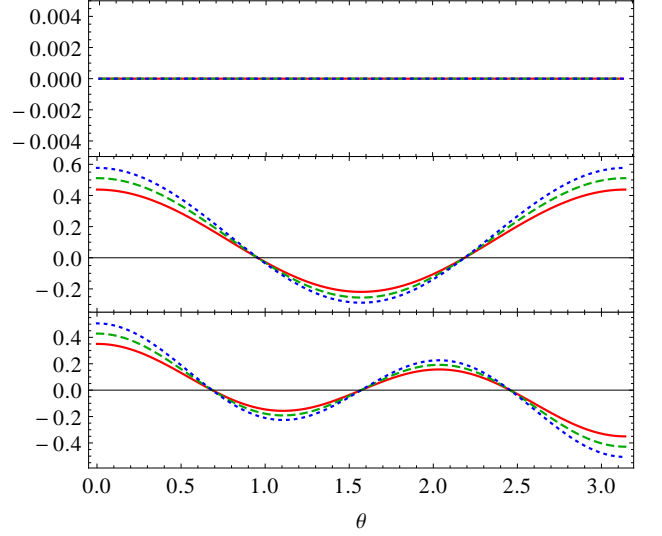


Figure 3: Total multipolar contribution from super-Hubble waves to the CMB as a function of  $\theta$ . From top to bottom we show the dipole  $\Delta T_1$ , quadrupole  $\Delta T_2$  and octopole  $\Delta T_3$ . Continuous, dashed and dotted curves represent the values  $n_s = (0.5, 1.0, 1.5)$ , respectively. The dipole is exactly zero in a  $\Lambda$ CDM universe; see the text for details.

The nominal value of the best fit quadrupole measured by Planck is

$$\frac{2(2+1)}{2\pi} C_2 = 299.495 \times 10^{-12}.$$

In the (unlikely) event that the full quadrupole is attributable to a super-Hubble wave as defined here, we obtain as a lower bound  $L/x_{\text{dec}} \gtrsim 120$ . Any fraction of the measured quadrupole smaller than one would only raise this value, so this is actually a conservative lower bound. In fact, since this ratio decreases with increasing  $C_2$ , a more conservative bound would be given by the measured  $C_2$  with error bars. The Planck team has shown that, at 68% confidence level, the quadrupole is

$$\frac{2(2+1)}{2\pi} C_2 = (299.495^{+797.980}_{-159.596}) \times 10^{-12}.$$

Taking the upper error bar, we find

$$\frac{L}{x_{\text{dec}}} \gtrsim 87. \quad (32)$$

Just for comparison, let us see what would a purely theoretical estimation look like. Using CAMB [21] to evaluate the quadrupole for the fiducial parameters used in this work, we find  $2(2+1)C_2/2\pi \simeq 1136 \times 10^{-12}$ . This gives  $L/x_{\text{dec}} \gtrsim 86$ , which is essentially the same as Eq. (32). Adding  $1\sigma$  cosmic variance error bar to this value will lower it to  $L/x_{\text{dec}} \gtrsim 76$ , which is again not too far from the observational bound. Interestingly, this shows that the tension between the measured and predicted value of the quadrupole  $C_2$  is not so drastic when viewed from

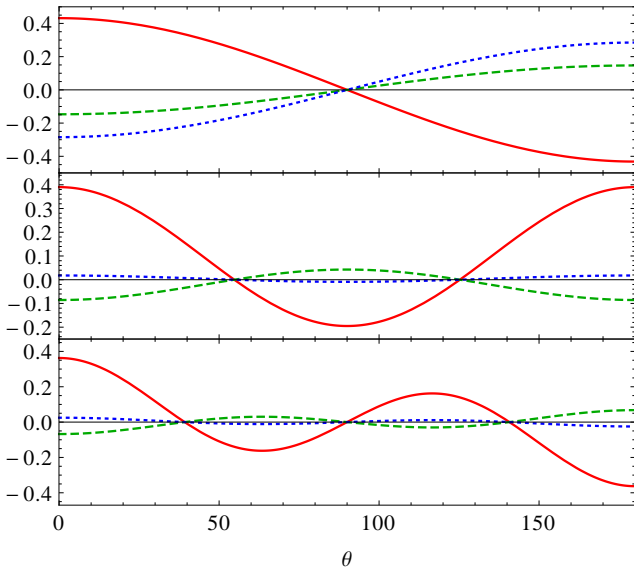


Figure 4: Individual multipolar contribution from the SW (continuous), ISW (dotted) and Doppler (dashed) effects as functions of  $\theta$ . A simple inspection shows that the sum of the curves in the first panel gives zero. See the first panel in Fig. (3). These plots were generated for a Harrison-Zel'dovich spectrum and show, from top to bottom, the dipole, quadrupole and octopole of each term in Eq. (30).

the real space perspective. This sort of analysis reinforces the idea that a real space approach to CMB [3, 4, 10] deserves attention, even if in principle it contains the same information as found in the more popular harmonic approach.

What does the bound in Eq. (32) have to say regarding the duration of inflation? Supposing that inflation has lasted  $N_{\min} = \ln a_f/a_i$  e-foldings, just enough to solve the horizon and flatness problems, the size of the homogeneous patch larger than the Hubble horizon at the onset of inflation would be only slightly larger than our horizon today. Therefore, any departure from the lower limit  $L/x_{\text{dec}} \gtrsim 1$  would imply in a lower bound to the number of e-foldings, that is  $N \gtrsim N_{\min} + \ln 87 \approx N_{\min} + 4$ . Incidentally, this suggests that CMB data is barely at the observational window for signatures of primordial anisotropies [22].

Finally, going back to the original question raised in this work, we can check that the bounds on the size of the homogeneous universe is a function of the scalar spectral index. Since red spectra are more sensitive to larger waves, the CMB quadrupole saturates at smaller super-Hubble lengths than the saturation length of the scale invariant (Harrison-Zel'dovich) spectrum. Blue spectra, on the other hand, are less sensitive to larger waves, making the measured quadrupole value saturate at larger super-Hubble lengths. This behavior is depicted in Fig. (5).

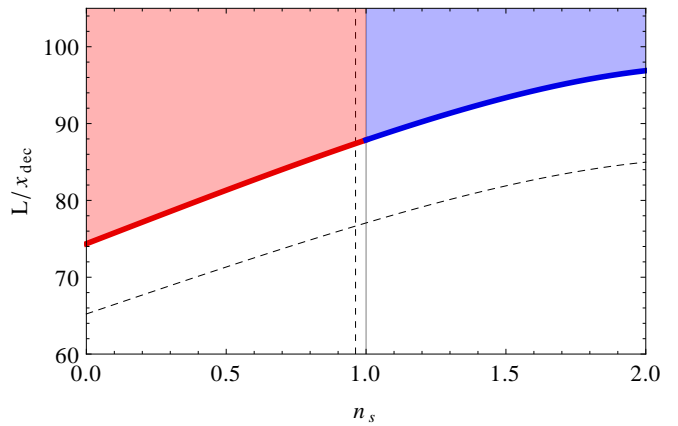


Figure 5: Quadrupolar bound on the ratio  $L/x_{\text{dec}}$  as a function of  $n_s$  based on Planck data with  $1\sigma$  (lower) error bar (see the text for details). Models with red spectrum ( $n_s < 1$ ) are more sensible to super-Hubble waves, implying in a smaller lower limit to the size of the universe. Blue spectrum ( $n_s > 1$ ) models give less weight to larger waves, implying in a higher lower bound. Vertical lines represent the Harrison-Zel'dovich (continuous) and Planck's best fit (dashed) spectra. The lower dashed line represent the  $\Lambda$ CDM model with  $1\sigma$  (lower) cosmic variance error bar.

## V. FINAL REMARKS

The primordial power spectrum predicted by most inflationary models displays a filtering property noticeable at very large cosmological scales. This can be demonstrated by writing the explicit dependence of the angular two-point correlation function with the scalar spectral index  $n_s$ . By employing the Sachs-Wolfe effect, we have shown that this filtering property produces a correlation function which is always positive if  $n_s > 1$ , whereas large scale correlations will unavoidably change sign if  $n_s < 1$ . This mechanism implies that the spectral index plays an important role in estimating the size of the homogeneous patch of the universe through, for example, the Grishchuk-Zel'dovich effect, where the size of the universe is estimated through the modulation it introduces at low CMB multipoles. In this work we have extended the GZ effect to comply with stochastic, super-Hubble waves. Because these waves have amplitudes which are filtered by the primordial power spectrum, the estimates on the allowed size of the largest super-Hubble wave depend on the value of the scalar spectral index  $n_s$ . Inflationary models predicting an index  $n_s < 1$  have a spectrum which is more sensitive to super-Hubble waves as compared to models predicting  $n_s > 1$ . This means that the contribution of red waves to the low- $\ell$  CMB multipoles will saturate at shorter super-Hubble length than the contribution of blue ones, implying in lower bounds to the size of the homogeneous patch of the universe. Using Planck's best fit value for  $n_s$  and the quadrupole  $C_2$ , we have found that the homogeneous universe extends to a distance at least 87 times bigger than our present optical

horizon. This result is compatible with analytical estimates found in previous works [12, 13, 15], and one order of magnitude smaller than the statistical analysis carried in [23], where the standard GZ effect was employed. However, we emphasize that our approach is based on a different philosophy. Instead of evoking a phenomenological sharp feature in the primordial power spectrum, we use the very inflationary power spectrum as cosmologically motivated spectral filter. In other words, we have measured the size of the universe from its interior up to a saturation quadrupole value fixed by CMB data. As one last comment, we would like to point to some possible applications of the results presented here. Recently, the authors of Ref. [24] claimed that frequency distortions on the blackbody spectrum of CMB, such as those produced by the kinetic Sunyaev-Zel'dovich effect and the Compton- $y$  distortion, could be used as complementary

data to the GZ effect. Indeed, in the light of our results, complementary information on the low- $\ell$  properties of the universe could be used to independently constrain the scalar spectral index, thus narrowing the space of cosmological parameters yet in the regime of Gaussian and isotropic perturbations.

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- [1] Planck Collaboration, P. Ade *et al.*, arXiv:1303.5076.
  - [2] Planck collaboration, P. Ade *et al.*, arXiv:1303.5075.
  - [3] S. Bashinsky and E. Bertschinger, Phys.Rev. **D65**, 123008 (2002), [astro-ph/0202215].
  - [4] L. R. Abramo, P. H. Reimberg and H. S. Xavier, Phys.Rev. **D82**, 043510 (2010), [1005.0563].
  - [5] A. Yoho, C. J. Copi, G. D. Starkman and T. S. Pereira, arXiv:1211.6756.
  - [6] L. R. Abramo and T. S. Pereira, Adv.Astron. **2010**, 378203 (2010), [1002.3173].
  - [7] WMAP Collaboration, D. Spergel *et al.*, Astrophys.J.Suppl. **148**, 175 (2003), [astro-ph/0302209].
  - [8] Planck Collaboration, P. Ade *et al.*, arXiv:1303.5083.
  - [9] C. Copi, D. Huterer, D. Schwarz and G. Starkman, Phys.Rev. **D75**, 023507 (2007), [astro-ph/0605135].
  - [10] C. J. Copi, D. Huterer, D. J. Schwarz and G. D. Starkman, Adv.Astron. **2010**, 847541 (2010), [1004.5602].
  - [11] L. Grishchuk and I. Zeldovich, Soviet Astronomy **22**, 125 (1978).
  - [12] V. Mukhanov and G. Chibisov, Soviet Astronomy Letters **10**, 374 (1984).
  - [13] A. L. Erickcek, S. M. Carroll and M. Kamionkowski, Phys.Rev. **D78**, 083012 (2008), [0808.1570].
  - [14] D. H. Lyth and A. Woszczyna, Phys.Rev. **D52**, 3338 (1995), [astro-ph/9501044].
  - [15] M. S. Turner, Phys.Rev. **D44**, 3737 (1991).
  - [16] T. S. Pereira, C. Pitrou and J.-P. Uzan, JCAP **0709**, 006 (2007), [0707.0736].
  - [17] T. S. Pereira, S. Carneiro and G. A. M. Marugan, JCAP **1205**, 040 (2012), [1203.2072].
  - [18] P. Peter and J.-P. Uzan, *Primordial cosmology* Oxford Graduate Texts (Oxford Univ. Press, Oxford, 2009).
  - [19] S. Weinberg, Phys.Rev. **D67**, 123504 (2003), [astro-ph/0302326].
  - [20] R. Sachs and A. Wolfe, Astrophys.J. **147**, 73 (1967).
  - [21] A. Lewis, A. Challinor and A. Lasenby, Astrophys.J. **538**, 473 (2000), [astro-ph/9911177].
  - [22] C. Pitrou, T. S. Pereira and J.-P. Uzan, JCAP **0804**, 004 (2008), [0801.3596].
  - [23] P. G. Castro, M. Douspis and P. G. Ferreira, Phys.Rev. **D68**, 127301 (2003), [astro-ph/0309320].
  - [24] P. Bull and M. Kamionkowski, arXiv:1302.1617.